On the canonical ring of curves and surfaces *

Marco Franciosi

Abstract

Let C be a curve (possibly non reduced or reducible) lying on a smooth algebraic surface. We show that the canonical ring $R(C,\omega_C)=\bigoplus_{k\geq 0}H^0(C,{\omega_C}^{\otimes k})$ is generated in degree 1 if C is numerically 4-connected, not hyperelliptic and even (i.e. with ω_C of even degree on every component).

As a corollary we show that on a smooth algebraic surface of general type with $p_g(S)\geq 1$ and $q(S)=0$ the canonical ring $R(S,K_S)$ is generated in degree ≤ 3 if there exists a curve $C\in|K_S|$ numerically 3-connected and not hyperelliptic.

keyword: algebraic curve, surface of general type, canonical ring, pluricanonical embedding.

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1 Introduction

Let C be a curve (possibly non reduced or reducible) lying on a smooth algebraic surface S and let ω_C be the dualizing sheaf of C. The purpose of this paper is to analyze the canonical ring of C, that is, the graded ring

$$R(C,\omega_C)=\bigoplus_{k\geq 0}H^0(C,{\omega_C}^{\otimes k})$$

under some suitable assumptions on the curve C.

The rationale of our analysis stems from several aspects of the theory of algebraic surfaces.

The first such aspect is the analysis of surface's fibrations and the study of their applications to surface's geography. Indeed, given a genus g fibration $f:S\to B$ over a smooth curve B, an important tool in this analysis is the *relative canonical algebra* $R(f)=\bigoplus_{n\geq 0}f_*(\omega_{S/B}^{\otimes n})$.

In recent years the importance of $R(f)$ has become clear (see Reid's unpublished note [18]) and a way to understand its behavior consists in studying the canonical ring

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of every fibre of f. More specifically, denoting by $C = f^{-1}(P)$ the scheme theoretic fibre over a point $p \in B$, the local structure around P of the *relative canonical algebra* can be understood via the canonical algebra of C, since the reduction modulo $\mathscr{M}_P$ of the stalk at P of the *relative canonical algebra* is nothing but $R(C,\omega_C)$ (see [18, §1]). Mendes Lopes in [16] studied the cases where the genus g of the fibre is $g \leq 3$ whereas in [14] and [11] it is shown that for every $g \geq 3$, $R(C,\omega_C)$ is generated in degree ≤ 4 if every fibre is numerically connected and in degree ≤ 3 if furthermore there are no multiple fibres.

More recently Catanese and Pignatelli in [7] illustrated two structure theorems for fibration of low genus using a detailed description of the relative canonical algebra. In particular they showed an interesting characterization of isomorphism classes of relatively minimal fibration of genus 2 and of relatively minimal fibrations of genus 3 with fibres numerically 2-connected and not hyperelliptic (see [7, Thms. 4.13, 7.13]).

Another motivation of our analysis (see [15]) relies on the study of the resolution of a normal surface singularity $\pi : S \to X$. Indeed if one considers the fundamental cycle arising from a minimal resolution of a singularity, then the study of the ring $R(C,K_S) = \bigoplus_{k\geq 0} H^0(C,{K_S}^{\otimes k})$ plays a crucial role in order to understand that singularity.

Finally, as shown in [6], the study of invertible sheaves on curves possibly reducible or non reduced is rich in implications in the cases where Bertini's theorem does not hold or simply if one needs to consider every curve contained in a given linear system. For instance, one can acquire information on the canonical ring of a surface of general type simply by taking its restriction to an effective canonical divisor $C \in |K_S|$ (not necessarily irreducible, neither reduced) and considering the canonical ring $R(C,\omega_C)$ (see Thm. 1.2 below).

In this paper we analyze the canonical ring of C when the curve C is m-connected and *even*, and we show some applications to the study of the canonical ring of an algebraic surface of general type.

For a curve C lying on a smooth algebraic surface S, being *m-connected* means that $C_1 \cdot C_2 \geq m$ for every effective decomposition $C = C_1 + C_2$, (where $C_1 \cdot C_2$ denotes their intersection number as divisor on S). If C is 1-connected usually C is said to be numerically connected. The definition turns back to Franchetta (cf. [10]) and has many relevant implications. For instance in [6, §3] it is shown that if the curve C is 1-connected then $h^0(C,\mathscr{O}_C) = 1$, if C is 2-connected then the system $|\omega_C|$ is base point free, whereas if C is 3-connected and not honestly hyperelliptic (i.e., a finite double cover of $\mathbb{P}^1$ induced by the canonical morphism) then ω_C is very ample.

Keeping the usual notation for effective divisor on smooth surfaces, i.e., writing C as $\sum_{i=1}^{s} n_i\Gamma_i$ (where Γ_i's are the irreducible components of C and n_i the multiplicity of Γ_i in C), the second condition can be illustrated by the following definition.

Definition *Let* $C = \sum_{i=1}^{s} n_i\Gamma_i$ *be a curve contained in a smooth algebraic surface. C is said to be* even *if* $\deg(\omega_{C|\Gamma_i})$ *is even for every irreducible* $\Gamma_i \subset C$ *(that is,* $\Gamma_i \cdot (C-\Gamma_i)$ *even for every* $i = 1,\dots,s$.*)*

Even curves appear for instance when considering the canonical system $|K_S|$ for a

surface S of general type. Indeed, by adjunction, for every curve $C \in |K_S|$ we have $|(2K_S)_{|C}| = |K_C|$, that is, every curve in the canonical system is even.

The main result of this paper is a generalization to even curves of the classical Theorem of Noether and Enriques on the degree of the generators of the graded ring $R(C, \omega_C)$:

Theorem 1.1 *Let C be an even 4-connected curve contained in a smooth algebraic surface. If $p_a(C) \geq 3$ and C is not honestly hyperelliptic then $R(C, \omega_C)$ is generated in degree 1.*

Following the notations of [17], this result can be rephrased by saying that ω_C is normally generated on C. In this case the embedded curve $\varphi_{|\omega_C|}(C) \subset \mathbb{P}^{p_a(C)-1}$ is arithmetically Cohen–Macaulay.

The proof of Theorem 1.1 is based on the ideas adopted by Mumford in [17] and on the results obtained in [11] for adjoint divisors, via a detailed analysis of the possible decompositions of the given curve C.

As a corollary we obtain a bound on the degree of the generators of the canonical ring of a surface of general type.

If S is a smooth algebraic surface and K_S a canonical divisor, the canonical ring of S is the graded algebra

$$R(S, K_S) = \bigoplus_{k \geq 0} H^0(S, {K_S}^{\otimes k})$$

In [8] a detailed analysis of $R(S, K_S)$ is presented in the most interesting case where S is of general type and there are given bounds (depending on the invariants $p_g(S) := h^0(S, K_S)$, $q := h^1(S, \mathcal{O}_S)$, and K_S^2) on the degree of elements of $R(S, K_S)$ forming a minimal system of homogeneous generators. Furthermore it is shown that for small values of p_g some exceptions do occur, depending substantially on the numerical connectedness of the curves in the linear system $|K_S|$. In particular in [8, §4] there are given examples of surfaces of general type with K_S not 3-connected whose canonical ring is not generated in degree ≤ 3 and it is conjectured that the 3-connectedness of the canonical divisor K_S should imply the generation of $R(S, K_S)$ in degree 1,2,3, at least in the case $q = 0$.

Here we show that this is the case. Our result, obtained essentially by restriction to a curve $C \in |K_S|$, is the following

Theorem 1.2 *Let S be a surface of general type with $p_g(S) := h^0(S, K_S) \geq 1$ and $q := h^1(S, \mathcal{O}_S) = 0$. Assume that there exists a curve $C \in |K_S|$ such that C is numerically 3-connected and not honestly hyperelliptic. Then the canonical ring of S is generated in degree ≤ 3.*

The paper is organized as follows: in §2 some useful background results are illustrated; in §3 we point out a result on even divisors; in §4 we consider the special case of binary curves, i.e. the case where C is the union of two rational curves; in §5 we prove Thm. 1.1; in §6 we give the proof of Thm. 1.2.

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2 Notation and preliminary results

2.1 Notation

We work over an algebraically closed field $\mathbb{K}$ of characteristic ≥ 0.

Throughout this paper S will be a smooth algebraic surface over $\mathbb{K}$ and C will be a curve lying on S (possibly reducible and non reduced). Therefore C will be written (as a divisor on S) as $C = \sum_{i=1}^{s} n_i\Gamma_i$, where the Γ_i's are the irreducible component of C and the n_i's are the multiplicities. A subcurve $B \subseteq C$ will mean a curve $\sum m_i\Gamma_i$, with $0 \leq m_i \leq n_i$ for every i.

By abuse of notation if $B \subset C$ is a subcurve of C, $C - B$ denotes the curve A such that $C = A + B$ as divisors on S.

Let $\mathscr{F}$ be an invertible sheaf on C.

If $G \subset C$ is a proper subcurve of C then $\mathscr{F}_{|G}$ denotes its restriction to G.

For each i the natural inclusion map $\varepsilon_i : \Gamma_i \to C$ induces a map $\varepsilon_i^* : \mathscr{F} \to \mathscr{F}_{|\Gamma_i}$. We denote by $d_i = \deg(\mathscr{F}_{|\Gamma_i}) = \deg_{\Gamma_i}\mathscr{F}$ the degree of $\mathscr{F}$ on each irreducible component, and by $\mathbf{d} := (d_1, ..., d_s)$ the multidegree of $\mathscr{F}$ on C. If $B = \sum m_i\Gamma_i$ is a subcurve of C, by $\mathbf{d}_B$ we mean the multidegree of $\mathscr{F}_{|B}$.

By $\mathrm{Pic}^{\mathbf{d}}(C)$ we denote the Picard scheme which parametrizes the classes of invertible sheaves of multidegree $\mathbf{d} = (d_1, \dots, d_s)$ (see [11]).

We recall that for every $\mathbf{d} = (d_1, ..., d_s)$ there is an isomorphism $\mathrm{Pic}^{\mathbf{d}}(C) \cong \mathrm{Pic}^{\mathbf{0}}(C)$ and furthermore $\dim \mathrm{Pic}^{\mathbf{0}}(C) = h^1(C, \mathscr{O}_C)$ (cf. e.g. [3]).

Concerning the Picard group of C and the Picard group of a subcurve $B \subset C$ we have

$$\mathrm{Pic}^{\mathbf{d}}(C) \twoheadrightarrow \mathrm{Pic}^{\mathbf{d}_B}(B) \quad \forall \mathbf{d}$$

(see [11, Rem. 2.1]).

An invertible sheaf $\mathscr{F}$ is said to be NEF if $d_i \geq 0$ for every i. Two invertible sheaves $\mathscr{F}$, $\mathscr{F}'$ are said to be numerically equivalent on C (notation: $\mathscr{F} \overset{\text{num}}{\sim} \mathscr{F}'$) if $\deg_{\Gamma_i}\mathscr{F} = \deg_{\Gamma_i}\mathscr{F}'$ for every $\Gamma_i \subseteq C$.

ω_C denotes the dualizing sheaf of C (see [13], Chap. III, §7), and $p_a(C)$ the arithmetic genus of C, $p_a(C) = 1 - \chi(\mathscr{O}_C)$.

If $G \subset C$ is a proper subcurve of C we denote by $H^0(G, \omega_C)$ the space of sections of $\omega_{C|G}$.

Finally, a curve C is said to be *honestly hyperelliptic* if there exists a finite morphism $\psi \colon C \to \mathbb{P}^1$ of degree 2. In this case C is either irreducible, or of the form $C = \Gamma_1 + \Gamma_2$ with $p_a(\Gamma_i) = 0$ and $\Gamma_1 \cdot \Gamma_2 = p_a(C) + 1$ (see [6, §3] for a detailed treatment).

2.2 General divisors of low degree

Let $C=\sum_{i=1}^{s} n_i\Gamma_i$ be a curve lying on a smooth algebraic surface S. An invertible sheaf on C of multidegree $\mathbf{d}=(d_1,\dots,d_s)$ is said to be "general" if the corresponding class in the Picard scheme $\mathrm{Pic}^{\mathbf{d}}(C)$ is in general position, i.e., if it lies in the complementary of a proper closed subscheme (see [11] for details).

We recall two vanishing results for general invertible sheaves of low degree.

Theorem 2.1 ([11, Thms. 3.1, 3.2]) *(i) If $\mathscr{F}$ is a "general" invertible sheaf such that $\deg_B\mathscr{F}\geq\ p_a(B)$ for every subcurve $B\subseteq C$, then it is $H^1(C,\mathscr{F})=0$.*

(ii) If $\mathscr{F}$ is a "general" invertible sheaf such that $\deg_B\mathscr{F}\geq\ p_a(B)+1$ for every subcurve $B\subseteq C$, then the linear system $|\mathscr{F}|$ is base point free.

In particular we obtain the following

Proposition 2.2 *Let $C=\sum_{i=1}^{s} n_i\Gamma_i$ be a 1-connected curve contained in a smooth algebraic surface, and consider a proper subcurve $B\subsetneq C$. Let $\mathbf{d}=(d_1,\dots,d_s)\in\mathbb{Z}^s$ be such that $d_i\geq\frac{1}{2}\deg_{\Gamma_i}\omega_C\ \forall\ i=1,\dots,s$.*

Then for a "general" invertible sheaf $\mathscr{F}$ in $\mathrm{Pic}^{\mathbf{d}_B}(B)$ it is:

(i) $H^1(B,\mathscr{F})=0$;

(ii) $|\mathscr{F}_{|B}|$ is a base point free system on B if C is 3- connected.

Considering the case where C is an even 4-connected curve we obtain

Corollary 2.3 *Let $C=\sum_{i=1}^{s} n_i\Gamma_i$ be a 4-connected even curve contained in a smooth algebraic surface.*

For every $i=1,\dots,s$, let $d_i=\frac{1}{2}\deg_{\Gamma_i}\omega_C$ and let $\mathbf{d}=(d_1,\dots,d_s)\in\mathbb{Z}^s$.

Let $B\subsetneq C$ be a proper subcurve of C and consider a a "general" invertible sheaf $\mathscr{F}$ in $\mathrm{Pic}^{\mathbf{d}_B}(B)$ (i.e., with an abuse of notation we can write $\mathscr{F}\overset{\text{num}}{\sim}\frac{1}{2}\omega_{C|_B}$).

Then $H^1(B,\mathscr{F})=0$ and $|\mathscr{F}_{|B}|$ is a base point free system.

2.3 Koszul cohomology groups of algebraic curves

Let $C=\sum_{i=1}^{s} n_i\Gamma_i$ be a curve lying on a smooth algebraic surface S and let $\mathscr{H},\ \mathscr{F}$ be invertible sheaves on C. Consider a subspace $W\subseteq H^0(C,\mathscr{F})$ which yields a base point free system of projective dimension r.

The Koszul groups $\mathscr{K}_{p,q}(C,W,\mathscr{H},\mathscr{F})$ are defined as the cohomology at the middle of the complex

$$\bigwedge^{p+1} W\otimes H^0(\mathscr{H}\otimes\mathscr{F}^{q-1})\longrightarrow\bigwedge^{p} W\otimes H^0(\mathscr{H}\otimes\mathscr{F}^{q})\longrightarrow\bigwedge^{p-1} W\otimes H^0(\mathscr{H}\otimes\mathscr{F}^{q+1})$$

If $W=H^0(C,\mathscr{F})$ they are usually denoted by $\mathscr{K}_{p,q}(C,\mathscr{H},\mathscr{F})$, while if $\mathscr{H}\cong\mathscr{O}_C$ the usual notation is $\mathscr{K}_{p,q}(C,\mathscr{F})$ (see [12] for the definition and main results).

We point out that the multiplication map

$$W\otimes H^0(C,\mathscr{H})\to H^0(C,\mathscr{F}\otimes\mathscr{H})$$

is surjective if and only if $\mathcal{K}_{0,1}(C,W,\mathcal{H},\mathcal{F})=0$ and the ring $R(C,\mathcal{F})=\bigoplus_{k\geq 0}H^0(C,\mathcal{F}^{\otimes k})$ is generated in degree 1 if and only if $\mathcal{K}_{0,q}(C,\mathcal{F})=0\ \forall\ q\geq 1$. Moreover if $\mathcal{F}$ is very ample and $R(C,\mathcal{F})$ is generated in degree 1, then, identifying C with its image in $\mathbb{P}^r\cong\mathbb{P}(H^0(\mathcal{F})^\vee)$, it is $\mathcal{K}_{1,1}(C,\mathcal{F})\cong I_2(C,\mathbb{P}^r)$, the space of quadrics in $\mathbb{P}^r$ vanishing on C (see [12]).

For our analysis the main applications of Koszul cohomology are the following propositions (see [11, §1], [14, §1] for further details on curves lying on smooth surfaces).

Proposition 2.4 (Duality) *Let $\mathcal{F}$, $\mathcal{H}$ be invertible sheaves on C and assume $W\subseteq H^0(C,\mathcal{F})$ to be a subspace of* $\dim=r+1$ *which yields a base point free system. Then*

$$\mathcal{K}_{p,q}(C,W,\mathcal{H},\mathcal{F})\overset{\mathrm{d}}{=}\mathcal{K}_{r-p-1,2-q}(C,W,\omega_C\otimes\mathcal{H}^{-1},\mathcal{F})$$

(where $\overset{\mathrm{d}}{=}$ means duality of vector space).

For a proof see [11, Prop. 1.4]. Following the ideas outlined in [14, Lemma 1.2.2] we have a slight generalization of Green's H^0-Lemma.

Proposition 2.5 (H^0-Lemma) *Let C be 1-connected and let $\mathcal{F}$, $\mathcal{H}$ be invertible sheaves on C and assume $W\subseteq H^0(C,\mathcal{F})$ to be a subspace of* $\dim=r+1$ *which yields a base point free system. If either*

(i) $H^1(C,\mathcal{H}\otimes\mathcal{F}^{-1})=0$,
or

(ii) C is numerically connected, $\omega_C\cong\mathcal{H}\otimes\mathcal{F}^{-1}$ and $r\geq 2$,
or

(iii) C is numerically connected, $h^0(C,\omega_C\otimes\mathcal{H}^{-1}\otimes\mathcal{F})\leq r-1$ and there exists a reduced subcurve $B\subseteq C$ such that:

- $W\cong W_{|B}$,
- $H^0(C,\omega_C\otimes\mathcal{H}^{-1}\otimes\mathcal{F})\hookrightarrow H^0(B,\omega_C\otimes\mathcal{H}^{-1}\otimes\mathcal{F})$,
- *every non–zero section of $H^0(C,\omega_C\otimes\mathcal{H}^{-1}\otimes\mathcal{F})$ does not vanish identically on any component of B;*

then $\mathcal{K}_{0,1}(C,W,\mathcal{H},\mathcal{F})=0$, that is, the multiplication map

$$W\otimes H^0(C,\mathcal{H})\to H^0(C,\mathcal{F}\otimes\mathcal{H})$$

is surjective.

Proof. By duality we need to prove that $\mathcal{K}_{r-1,1}(C,W,\omega_C\otimes\mathcal{H}^{-1},\mathcal{F})=0$. To this aim let $\{s_0,\dots,s_r\}$ be a basis for W and let $\alpha=\sum s_{i_1}\wedge s_{i_2}\wedge\ldots\wedge s_{i_{r-1}}\otimes\alpha_{i_1i_2\ldots i_{r-1}}\in\bigwedge^{r-1}W\otimes H^0(C,\omega_C\otimes\mathcal{H}^{-1}\otimes\mathcal{F})$ be an element in the Kernel of the Koszul map $d_{r-1,1}$.

In cases (i) obviously $\alpha=0$ since by Serre duality it is $H^0(C,\omega_C\otimes\mathcal{H}^{-1}\otimes\mathcal{F})\cong H^1(C,\mathcal{H}\otimes\mathcal{F}^{-1})=0$. In case and (ii) it is $H^0(C,\omega_C\otimes\mathcal{H}^{-1}\otimes\mathcal{F})=H^0(C,\mathcal{O}_C)=\mathbb{K}$ by connectedness and we conclude similarly (see also [11, Prop. 1.5]).

In the latter case by our assumptions we can restrict to the curve B. Since B is reduced we can choose $r+1$ "sufficiently general points" on B so that $s_j(P_i) = \delta^i_j$. But then $\alpha \in \ker(d_{r-1,1})$ implies for every multiindex $\mathbf{I} = \{i_1, \dots i_{r-2}\}$ the following equation (up to sign)

$$\alpha_{j_1 i_1 \dots i_{r-2}} \cdot s_{j_1} + \alpha_{j_2 i_1 \dots i_{r-2}} \cdot s_{j_2} + \alpha_{j_3 i_1 \dots i_{r-2}} s_{j_3} = 0.$$

(where $\{i_1, \dots i_{r-2}\} \cup \{j_1, j_2, j_3\} = \{0, \dots, r+1\}$).

Evaluating at $P_j' s$ and reindexing we get $\alpha_{i_1 \dots i_{r-1}}(P_{i_k}) = 0$ for $k = 1, \dots, r-1$.

Let $\tilde{r} = h^0(C, \omega_C \otimes \mathscr{H}^{-1} \otimes \mathscr{F})$. Since the $P_j' s$ are in general positions and every section of $H^0(C, \omega_C \otimes \mathscr{H}^{-1} \otimes \mathscr{F})$ does not vanish identically on any component of B, we may assume that any $(\tilde{r}-1)$-tuple of points $P_{i_1}, \dots, P_{i_{r-1}}$ imposes independent conditions on $H^0(C, \omega_C \otimes \mathscr{H}^{-1} \otimes \mathscr{F})$. The proposition then follows by a dimension count since by assumptions $\tilde{r} = h^0(C, \omega_C \otimes \mathscr{H}^{-1} \otimes \mathscr{F}) \leq h^0(C, \mathscr{F}) - 2 = r-1$.

$\square$

In some particular cases we can obtain deeper results, which will turn out to be useful for our induction argument in the proof of Theorem 1.1.

Proposition 2.6 *Let C be either*

(i) an irreducible curve of arithmetic genus $p_a(C) \geq 1$; or

(ii) a binary curve of genus ≥ 1, that is, $C = \Gamma_1 + \Gamma_2$, with Γ_i irreducible and reduced rational curves s.t. $\Gamma_1 \cdot \Gamma_2 = p_a(C) + 1 \geq 2$ (see §4 for details).

Let $\mathscr{H} \overset{\text{num}}{\sim} \omega_C \otimes \mathscr{A}$ be a very ample divisor on C s.t. $\deg_C \mathscr{A} \geq 4$.

Then $\mathscr{K}_{0,1}(C, \mathscr{H}, \omega_C) = 0$, that is $H^0(C, \omega_C) \otimes H^0(C, \mathscr{H}) \twoheadrightarrow H^0(C, \omega_C \otimes \mathscr{H})$.

Proof. If $p_a(C) = 1$ then under our assumptions it is $\omega_C \cong \mathscr{O}_C$, whence the theorem follows easily.

If $p_a(C) \geq 2$ then by [6, Thms. 3.3, 3.4] $|\omega_C|$ is b.p.f. and moreover it is very ample if C is not honestly hyperelliptic. We apply Prop. 2.5 with $\mathscr{F} = \omega_C$ and $W = H^0(\omega_C)$.

If C is irreducible and $h^0(C, \omega_C \otimes \mathscr{A}^{-1}) = 0$ then the result follows by (i) of Prop. 2.5. If $h^0(C, \omega_C \otimes \mathscr{A}^{-1}) \neq 0$ and $h^0(C, \mathscr{A}) = 0$ it follows by Riemann-Roch. In the remaining case we obtain $h^0(C, \omega_C \otimes \mathscr{A}^{-1}) \leq h^0(C, \omega_C) - 2$ by Clifford's theorem since $\deg_C \mathscr{A} \geq 4$.

If $C = \Gamma_1 + \Gamma_2$ and $p_a(C) \geq 2$ we consider firstly the case where $\deg_{\Gamma_i} \mathscr{A} \geq -1$ for $i = 1, 2$. Under this assumption any non-zero section of $H^0(C, \omega_C \otimes \mathscr{A}^{-1})$ does not vanish identically on any single component of C (otherwise it would yield a section in $H^0(\Gamma_i, \omega_{\Gamma_i} \otimes \mathscr{A}^{-1}) \cong H^0(\mathbb{P}^1, -\alpha)$ with $\alpha \geq 1$). Therefore we can proceed exactly as in the irreducible case.

Now assume $C = \Gamma_1 + \Gamma_2$, $\deg_{\Gamma_2} \mathscr{A} \leq -2$ and $\deg_{\Gamma_1} \mathscr{A} \geq 6$. In this case we can apply (iii) of Prop. 2.5 taking $B = \Gamma_2$. Indeed, it is $h^0(\Gamma_1, \omega_{\Gamma_1} \otimes \mathscr{A}^{-1}) = h^0(\Gamma_1, \omega_{\Gamma_1}) = 0$ and we have the following maps

$$H^0(C, \omega_C) \cong H^0(\Gamma_2, \omega_C) \ ; \ \ H^0(C, \omega_C \otimes \mathscr{A}^{-1}) \hookrightarrow H^0(\Gamma_2, \omega_C \otimes \mathscr{A}^{-1}).$$

To complete the proof it remains to show that $h^0(C,\omega_C\otimes\mathscr{A}^{-1})\le h^0(C,\omega_C)-2=p_a(C)-2$. This follows by the following exact sequence

$$0\to H^0(\Gamma_2,\omega_{\Gamma_2}\otimes\mathscr{A}^{-1})\to H^0(C,\omega_C\otimes\mathscr{A}^{-1})\to H^0(\Gamma_1,\omega_C\otimes\mathscr{A}^{-1})\to 0.$$

In fact if $\deg_{\Gamma_1}(\omega_C\otimes\mathscr{A}^{-1})\ge 0$ we have $h^0(C,\omega_C\otimes\mathscr{A}^{-1})=p_a(C)-1-\deg\mathscr{A}$, whereas it is $h^0(C,\omega_C\otimes\mathscr{A}^{-1})=h^0(\Gamma_2,\omega_{\Gamma_2}\otimes\mathscr{A}^{-1})=-\deg_{\Gamma_2}\mathscr{A}-1<p_a(C)-2$ if $\deg_{\Gamma_1}(\omega_C\otimes\mathscr{A}^{-1})<0$ since $\deg_{\Gamma_2}(\omega_C+\mathscr{A})\ge 1$ by the ampleness of $\omega_C\otimes\mathscr{A}$.

□

If one considers a curve C with many components another useful tool is the following long exact sequences for Koszul groups.

Proposition 2.7 *Let $C=A+B$ and let $|\mathscr{F}|$ be a complete base point free system on C such that $H^0(C,\mathscr{F})\twoheadrightarrow H^0(A,\mathscr{F})$, $H^0(C,\mathscr{F}^{\otimes k})\twoheadrightarrow H^0(B,\mathscr{F}^{\otimes k})$ for every $k\ge 2$ and $H^0(A,\mathscr{F}(-B))=0$. Then we have a long exact sequence*

$$\begin{array}{lllll}\cdots & \to & \mathscr{K}_{p+1,q-1}(C,\mathscr{F}) & \to \mathscr{K}_{p+1,q-1}(B,W,\mathscr{F}) & \to \mathscr{K}_{p,q}(A,\mathscr{O}_A(-B),\mathscr{F}) \\ & \to & \mathscr{K}_{p,q}(C,\mathscr{F}) & \to \mathscr{K}_{p,q}(B,W,\mathscr{F}) & \to \cdots\end{array}$$

where $W=H^0(C,\mathscr{F})_{|B}=\mathrm{im}\{H^0(C,\mathscr{F})\to H^0(B,\mathscr{F})\}$.

Proof. Let us consider

$$B^1=\bigoplus_{q\ge 0}H^0(A,\mathscr{F}^{\otimes q}(-B)),\ \ B^2=\bigoplus_{q\ge 0}H^0(C,\mathscr{F}^{\otimes q}),\ \ B^3=W\oplus(\bigoplus_{q\ne 1}H^0(B,\mathscr{F}^{\otimes q}))$$

By our hypotheses the above vector spaces can be seen as $S(H^0(C,\mathscr{F}))-$modules and moreover they fit into the following exact sequence

$$0\to\ B^1\to\ B^2\to\ B^3\to 0$$

where the maps preserve the grading. By the long exact sequence for Koszul Cohomology (cf. [12, Corollary 1.4.d, Thm. 3.b.1]) we can conclude.

□

Notice that if C is numerically connected, $\mathscr{F}\cong\omega_C$, B is numerically connected and A is the disjoint union of irreducible rational curves then the above hypotheses are satisfied.

We point out that if $\mathscr{F}$ is very ample but the restriction map $H^0(C,\mathscr{F})\to H^0(B,\mathscr{F})$ is not surjective then, following the notation of [1], we can talk of "Weak Property N_p" for the curve B embedded by the system $W=H^0(C,\mathscr{F})_{|B}$.

2.4 Divisors normally generated on algebraic curves

To conclude this preliminary section we recall a theorem proved in [11] which allows us to start our analysis.

Theorem 2.8 ([11, Thm. A]) *Let C be a curve contained in a smooth algebraic surface and let $\mathscr{H} \overset{\text{num}}{\sim} \mathscr{F} \otimes \mathscr{G}$, where $\mathscr{F}, \mathscr{G}$ are invertible sheaf such that*

$$\begin{array}{ll} \deg \mathscr{F}_{|B} \geq \ p_a(B)+1 & \forall \ \textit{subcurve} \ \ B \subseteq C \\ \deg \mathscr{G}_{|B} \geq \ p_a(B) & \forall \ \textit{subcurve} \ \ B \subseteq C \end{array}$$

Then for every $n \geq 1$ the natural multiplication map $(H^0(C, \mathscr{H}))^{\otimes n} \to H^0(C, \mathscr{H}^{\otimes n})$ is surjective.

Moreover, applying the same arguments used in [11, Proof of Thm. A, p. 327] we have

Proposition 2.9 *Let C be a curve contained in a smooth algebraic surface and let $\mathscr{H}_1$, $\mathscr{H}_2$ be two invertible sheaves such that $\mathscr{H}_1 \overset{\text{num}}{\sim} \mathscr{F} \otimes \mathscr{G}_1$, $\mathscr{H}_2 \overset{\text{num}}{\sim} \mathscr{F} \otimes \mathscr{G}_2$ with*

$$\begin{array}{ll} \deg \mathscr{F}_{|B} \geq \ p_a(B)+1 & \forall \ \textit{subcurve} \ \ B \subseteq C \\ \deg \mathscr{G}_{1|B} \geq \ p_a(B) & \forall \ \textit{subcurve} \ \ B \subseteq C \\ \deg \mathscr{G}_{2|B} \geq \ p_a(B) & \forall \ \textit{subcurve} \ \ B \subseteq C \end{array}$$

(this holds e.g. if $\mathscr{H}_2 \overset{\text{num}}{\sim} \mathscr{H}_1 \otimes \mathscr{A}$, with $\mathscr{A}$ a NEF invertible sheaf).

Then $H^0(C, \mathscr{H}_1) \otimes H^0(C, \mathscr{H}_2) \twoheadrightarrow H^0(C, \mathscr{H}_1 \otimes \mathscr{H}_2)$.

3 Even curves and even divisors

Let $C = \sum_{i=1}^{s} n_i \Gamma_i$ be a curve contained in a smooth algebraic surface.

C is said to be *even* if

$$\deg_{\Gamma_i} \omega_C \text{ is even for every irreducible } \ \Gamma_i \subset C$$

(see §1). Similarly, if $\mathscr{H}$ is an invertible sheaf on C, then $\mathscr{H}$ is said to be *even* if

$$\deg_{\Gamma_i} \mathscr{H} \text{ is even for every irreducible } \ \Gamma_i \subset C$$

For even invertible sheaves of high degree the normal generation follows easily:

Theorem 3.1 *Let $C = \sum_{i=1}^{s} n_i \Gamma_i$ be a curve contained in a smooth algebraic surface and let $\mathscr{H}$ be an even invertible sheaf on C such that*

$$\deg_B \mathscr{H} \geq \ 2p_a(B)+2 \quad \forall \ \textit{subcurve} \ \ B \subseteq C$$

Then for every $n \geq 1$ the natural multiplication map $(H^0(C, \mathscr{H}))^{\otimes n} \to H^0(C, \mathscr{H}^{\otimes n})$ is surjective.

Proof. First of all notice that $\mathscr{H}$ is very ample by [6, Thm. 1.1]. Moreover since $\mathscr{H}$ is even there exists an invertible sheaf $\mathscr{F}$ such that $\mathscr{F}^{\otimes 2} \overset{\text{num}}{\sim} \mathscr{H}$. By our numerical assumptions for every subcurve $B \subseteq C$ it is $\deg_B \mathscr{F} \geq \ p_a(B)+1$ and $\deg_B(\mathscr{H} \otimes \mathscr{F}^{-1}) \geq p_a(B)+1$, whence we can conclude by Thm. 2.8. $\square$

4 The canonical ring of a binary curve

In this section we deal with the particular case of binary curves, which is particularly interesting from our point of view since it shows many of the pathologies that may occur studying curves with many components.

Definition 4.1 *A curve C is said to be a* binary curve *if $C = \Gamma_1 + \Gamma_2$ with Γ_1, Γ_2 irreducible and reduced rational curves such that $\Gamma_1 \cdot \Gamma_2 = p_a(C) + 1$ (see* [2] *)*

If C is not honestly hyperelliptic and $p_a(C) \geq 3$, then ω_C is very ample on C by [6, Thm. 3.6] and $\varphi_{|\omega_C|}$ embeds C as the union of two rational normal curves intersecting in a 0-dimensional scheme of length = $p_a(C) + 1$.

The theory of Koszul cohomology groups allow us to give a deep analysis of the ideal ring of the embedded curves $\varphi_{|\omega_C|}(C) \subset \mathbb{P}^{p_a(C)-1}$. The first proposition, which we will use in the proof of our main theorem, is that $R(C, \omega_C)$ is generated in degree 1. We point out that in [4, Prop. 3] there is an alternative proof of this result, but we prefer to keep ours in order to show how the theory of Kuszul cohomology groups works.

Proposition 4.2 *Let C be a binary curve of genus ≥ 3, not honestly hyperelliptic. Then $R(C, \omega_C)$ is generated in degree 1.*

Proof. The proposition follows if we prove the vanishing of the Koszul groups $\mathscr{K}_{0,q}(C, \omega_C)$ $\forall\, q \geq 1$.

For simplicity let $r = p_a(C) - 1 = \deg \mathscr{O}_{\Gamma_i}(\omega_C)$ $(i = 1, 2)$ and let us identify Γ_1, Γ_2 with their images in $\mathbb{P}^r$. Notice that we have $H^0(C, \omega_C) \cong H^0(\Gamma_i, \omega_C)$ (for $i = 1, 2$), while it is $H^0(\Gamma_1, \omega_C(-\Gamma_2)) = 0$ and $H^1(\Gamma_1, \omega_C^{\otimes k}(-\Gamma_2)) = 0$ for every $k \geq 2$. Therefore we can apply Prop. 2.7 getting the following exact sequence of Koszul groups

$$\cdots \to \mathscr{K}_{1,q-1}(\Gamma_2, \omega_C) \to \mathscr{K}_{0,q}(\Gamma_1, \mathscr{O}_{\Gamma_1}(-\Gamma_2), \omega_C) \to \mathscr{K}_{0,q}(C, \omega_C) \twoheadrightarrow \mathscr{K}_{0,q}(\Gamma_2, \omega_C)$$

Now since $|\omega_C|$ embeds Γ_2 as a rational normal curve in $\mathbb{P}^r$ we have $\mathscr{K}_{0,q}(\Gamma_2, \omega_C) = 0$ for all $q \geq 1$. Moreover by [12, (2.a.17)] we have

$$\mathscr{K}_{0,q}(\Gamma_1, \mathscr{O}_{\Gamma_1}(-\Gamma_2), \omega_C) \cong \mathscr{K}_{0,0}(\Gamma_1, \omega_C^{\otimes q} \otimes \mathscr{O}_{\Gamma_1}(-\Gamma_2), \omega_C)$$

But on a rational curve we can easily determine the vanishing of the Koszul groups. Indeed, by [12, Corollary 3.a.6], if $\Gamma_1 \cong \mathbb{P}^1$, $\deg(\mathscr{F}) = d, \deg(\mathscr{H}) = a$ we have

$$\mathscr{K}_{p,0}(\Gamma_1, \mathscr{H}, \mathscr{F}) = 0 \;\text{ unless }\; 0 \leq a \leq 2d - 2 \text{ and } \; a - d + 1 \leq p \leq a.$$

Therefore a simple degree computation yields $\mathscr{K}_{0,q}(\Gamma_1, \mathscr{O}_{\Gamma_1}(-\Gamma_2), \omega_C) = 0$ for $q \geq 3$, which implies $\mathscr{K}_{0,q}(C, \omega_C) = 0$ $\forall\, q \neq 2$ by the above exact sequence of Koszul groups. If $q = 2$, it is $\mathscr{K}_{0,2}(\Gamma_1, \mathscr{O}_{\Gamma_1}(-\Gamma_2), \omega_C) \cong \mathscr{K}_{0,0}(\Gamma_1, \omega_C^{\otimes 2} \otimes \mathscr{O}_{\Gamma_1}(-\Gamma_2), \omega_C) \cong H^0(\Gamma_1, \omega_C^{\otimes 2} \mathscr{O}_{\Gamma_1}(-\Gamma_2)) \cong H^0(\Gamma_1, \omega_C \otimes \omega_{\Gamma_1}) \cong H^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(r-2))$. In particular it is $\neq 0$.

In this case, to show that $\mathscr{K}_{0,2}(C, \omega_C) = 0$ we are going to prove that the map $P : \mathscr{K}_{1,1}(\Gamma_2, \omega_C) \to \mathscr{K}_{0,2}(\Gamma_1, \mathscr{O}_{\Gamma_1}(-\Gamma_2), \omega_C)$ is surjective.

Notice that it is $\mathscr{K}_{1,1}(\Gamma_2, \omega_C) \cong I_2(\Gamma_2, \mathbb{P}^r)$, the space of quadrics in $\mathbb{P}^r$ vanishing on Γ_2, and it is well known that $I_2(\Gamma_2, \mathbb{P}^r) \cong \mathbb{K}^{\binom{r}{2}}$ (see e.g., [12, Thm. 3.c.6]). The

idea of the proof is to show that we can find $r-1$ points on Γ_1 imposing independent conditions on the quadrics in $I_2(\Gamma_2,\mathbb{P}^r)$. The proof will be given by induction on r.

For $r=2$ it is $\mathscr{K}_{1,1}(\Gamma_2,\omega_C) \twoheadrightarrow \mathscr{K}_{0,2}(\Gamma_1,\mathscr{O}_{\Gamma_1}(-\Gamma_2))$ since C is embedded as a plane quartic. For $r \geq 3$ take a point $Q \in \Gamma_1 \cap \Gamma_2$, choose coordinates $(x_0 : .. : x_r)$ so that $Q=(0:...:0:1)$ and consider the projection from Q onto the hyperplane $\{x_r=0\}$:

$$\begin{array}{rcl} \pi_Q : \mathbb{P}^r & \to & \mathbb{P}^{r-1} \\ (x_0 : ... : x_r) & \mapsto & (x_0 : ... : x_{r-1}) \end{array}$$

$\pi_Q(C) := \tilde{C} \subset \mathbb{P}^{r-1}$ is again a binary curve canonically embedded in $\mathbb{P}^{r-1}$ by the complete system $|\omega_{\tilde{C}}|$. The projection map restricted to C and to its subcurves Γ_1,Γ_2 induces via pull-back maps between Koszul groups of the same degree, which fit in the following commutative diagram

$$\begin{tikzcd}
\mathscr{K}_{1,1}(\tilde{C},\omega_{\tilde{C}}) \arrow[r] \arrow[d, hook, "\pi_Q^*"] & \mathscr{K}_{1,1}(\Gamma_2,\omega_{\tilde{C}}) \arrow[r, "P_1"] \arrow[d, hook, "\pi_{2,Q}^*"] & \mathscr{K}_{0,2}(\Gamma_1,\mathscr{O}_{\Gamma_1}(-\Gamma_2),\omega_{\tilde{C}}) \arrow[d, hook, "\pi_{1,Q}^*"] \\
\mathscr{K}_{1,1}(C,\omega_C) \arrow[r] \arrow[d, two heads] & \mathscr{K}_{1,1}(\Gamma_2,\omega_C) \arrow[r, "P"] \arrow[d, two heads] & \mathscr{K}_{0,2}(\Gamma_1,\mathscr{O}_{\Gamma_1}(-\Gamma_2),\omega_C) \arrow[d, two heads] \\
\mathscr{Q}(\omega_C) \arrow[r, hook] & \mathscr{Q}_2(\omega_C) \arrow[r, "P_2"] & \mathscr{R}
\end{tikzcd}$$

where $\mathscr{Q}(\omega_C)$ (resp. $\mathscr{Q}_2(\omega_C)$, $\mathscr{R}$) denotes the cokernel of π_Q^* (resp. the cokernel of ${\pi_{2,Q}}^*$, ${\pi_{1,Q}}^*$), and where we have identified the curves Γ_1 and Γ_2 with their embeddings.

By our construction $\pi_{2,Q}^*(\mathscr{K}_{1,1}(\Gamma_2,\omega_{\tilde{C}})) \subset \mathscr{K}_{1,1}(\Gamma_2,\omega_C)$ is the subspace spanned by the equations of the cones with center Q over the corresponding quadric in $I_2(\Gamma_2,\mathbb{P}^{r-1}) \cong \mathscr{K}_{1,1}(\Gamma_2,\omega_{\tilde{C}})$. Moreover $\mathscr{Q}_2(\omega_C)$ is generated by the following equations

$$x_i x_r - x_{i+1}x_{r-1} \qquad \text{for } i=0,...,r-2.$$

(see e.g [9, Ch. 6 , Prop. 6.1]).

To conclude we choose a 0-dimensional scheme A of length $=r-1$ so that $A' = A \cap \{x_r=0\}$ is a scheme of length $=r-2$ and $\mathscr{R} \cong \mathscr{I}_{A'}/\mathscr{I}_A \cong \mathbb{K}_{Q'}$ with $Q' \in \Gamma_1$ in general position. Since Γ_1 is rational it is immediately seen that

$$\mathscr{K}_{0,2}(\Gamma_1,\mathscr{O}_{\Gamma_1}(-\Gamma_2),\omega_C) \cong H^0(\Gamma_1,\omega_C \otimes \omega_{\Gamma_1}) \cong \mathscr{O}_A$$

and similarly $\mathscr{K}_{0,2}(\Gamma_1,\mathscr{O}_{\Gamma_1}(-\Gamma_2),\omega_{\tilde{C}}) \cong \mathscr{O}_{A'}$. Now, by induction hypothesis P_1 is surjective, as well as P_2 since Q' is general.

Therefore we can conclude that $P : \mathscr{K}_{1,1}(\Gamma_2,\omega_C) \to \mathscr{K}_{0,2}(\Gamma_1,\mathscr{O}_{\Gamma_1}(-\Gamma_2),\omega_C)$ is onto, which is what we wanted to prove. $\square$

5 Disconnecting components of numerically connected curves

Taking an an irreducible component $\Gamma \subset C$ one problem is that the restriction map $H^0(C,\omega_C) \to H^0(\Gamma,\omega_{C|\Gamma})$ is not surjective if $h^0(C-\Gamma,\mathscr{O}_{C-\Gamma}) = h^1(C-\Gamma,\omega_{C-\Gamma}) \geq 2$.

Nevertheless, if there exists a curve Γ with this property, it plays a special role in the proof of our main result.

To be more explicit, let us firstly consider the natural notion of disconnecting subcurve.

Definition 5.1 *Let $C = \sum_{i=1}^{s} n_i\Gamma_i$ be a numerically connected curve. A subcurve $B \subset C$ is said to be a* disconnecting subcurve *if $h^0(C-B, \mathscr{O}_{C-B}) \geq 2$.*

If B is a disconnecting curve then by the exact sequence

$$H^0(C-B, \omega_{C-B}) \to H^0(C, \omega_C) \to H^0(B, \omega_C) \to H^1(C-B, \omega_{C-B}) \to H^1(C, \omega_C)$$

we deduce that the restriction map $H^0(C, \omega_C) \to H^0(B, \omega_{C|B})$ can not be surjective. In this case following the arguments pointed out in [15] and [14] one can consider an "intermediate" curve G such that $B \subseteq G \subseteq C$ and $H^0(C, \omega_C) \twoheadrightarrow H^0(G, \omega_{C|G})$. We have the following useful Lemma.

Lemma 5.2 *Let $C = \sum_{i=1}^{s} n_i\Gamma_i$ be a m-connected curve ($m \geq 1$) and $\Gamma \subset C$ be an irreducible and reduced disconnecting subcurve. Let G be a minimal subcurve of C such that $H^0(C, \omega_C) \twoheadrightarrow H^0(G, \omega_{C|G})$ and $\Gamma \subseteq G \subseteq C$.*

Setting $E := C - G$, $G' := G - \Gamma$, then

(a) E is a maximal subcurve of $C - \Gamma$ such that $h^1(E, \omega_E) = h^0(E, \mathscr{O}_E) = 1$;

(b) Γ is of multiplicity 1 in G, $\omega_G \otimes (\omega_C)^{-1} \cong \mathscr{O}_G(-E)$ is NEF on G';

(c) $\deg_\Gamma(E) = \deg_{G'}(-E) + e$ with $e \geq m$;

(d) $h^1(E+\Gamma, \omega_{E+\Gamma}) = 1$, hence $H^0(C, \omega_C) \twoheadrightarrow H^0(G', \omega_{C|G'})$;

(e) G is m-connected and in particular it is $h^1(G, \omega_G) = 1$;

Proof. By hypotheses $H^0(C, \omega_C) \not\twoheadrightarrow H^0(\Gamma, \omega_{C|\Gamma})$ and G is a minimal subcurve such that $H^0(C, \omega_C) \twoheadrightarrow H^0(G, \omega_{C|G})$. Therefore $E = C - G$ is a maximal subcurve of $C - \Gamma$ such that $h^1(E, \omega_E) = h^1(C, \omega_C) = 1$, proving (a).

Moreover by [14, Lemma 2.2.1] either $\omega_G \otimes (\omega_C)^{-1}$ is NEF on G, or Γ is of multiplicity one in G and $\omega_G \otimes (\omega_C)^{-1}$ is NEF on $G - \Gamma = G'$.

Now by adjunction it is $\omega_G \otimes (\omega_C)^{-1} \cong \mathscr{O}_G(-E)$, which has negative degree on G since C is numerically connected by assumption. Therefore we can exclude the first case and by [14, Lemma 2.2.1] we conclude that Γ is of multiplicity one in G, $\omega_G \otimes (\omega_C)^{-1} \cong \mathscr{O}_G(-E)$ is NEF on $G' := G - \Gamma$, and $\deg_\Gamma(E) = \deg_{G'}(-E) + e$ with $e \geq m$, proving (b) and (c).

To prove (d) consider the two curves E and Γ. It is $h^1(E, \omega_E) = 1$ by (a) and $h^1(\Gamma, \omega_{E+\Gamma}) = 0$ because $\deg_\Gamma(\omega_{E+\Gamma}) \geq 2p_a(\Gamma) - 1$, whence we conclude considering the exact sequence

$$H^1(E, \omega_E) \to H^1(E+\Gamma, \omega_{E+\Gamma}) \to H^1(\Gamma, \omega_{E+\Gamma}) = 0$$

(e) follows since $\mathscr{O}_{G'}(-E)$ is NEF on G'. In fact if $B \subset G$ without loss of generality we may assume $B \subset G'$, and then we obtain $B \cdot (G - B) = B \cdot (C - B) - E \cdot B \geq B \cdot (C -$

$B) \geq m$ since it is $G = C - E$, that is G is m-connected. $h^1(G, \omega_G) = 1$ follows by [6, Thm. 3.3].

$\square$

With an abuse of notation we will call a subcurve $E \subset C$ as in Lemma 5.2 a *maximal connected subcurve* of $C - \Gamma$.

The above Lemma allows us to consider the splitting $C = G + E$ since by connectedness both the restriction maps $H^0(C, \omega_C) \to H^0(G, \omega_{C|G})$ and $H^0(C, \omega_C) \to H^0(E, \omega_{C|E})$ are surjective.

Concerning the subcurve G we have the following theorem.

Theorem 5.3 *Let $C = \sum_{i=1}^{s} n_i \Gamma_i$ be an even 4-connected curve and assume there exists an irreducible and reduced disconnecting subcurve $\Gamma \subset C$.*

Let G be a minimal subcurve of C such that $H^0(C, \omega_C) \twoheadrightarrow H^0(G, \omega_C)$ and $\Gamma \subseteq G \subseteq C$.

Then on G the multiplication map $H^0(G, \omega_C) \otimes H^0(G, \omega_G) \to H^0(G, \omega_C \otimes \omega_G)$ is surjective.

To simplify the notation, for every subcurve $B \subset C$ by $H^0(B, \omega_C)$ we will denote the space of sections of $\omega_{C|B}$.

If there exists a disconnecting component Γ and a decomposition $C = G' + \Gamma + E$ as in Lemma 5.2 such that $h^1(G', \omega_{G'}) \geq 2$ then we need an auxiliary Lemma.

Lemma 5.4 *Let $C = \sum_{i=1}^{s} n_i \Gamma_i$ be an even 4-connected curve and assume there exists an irreducible and reduced disconnecting subcurve $\Gamma \subset C$.*

If there exists a decomposition $C = G' + \Gamma + E$ as in Lemma 5.2 such that $h^1(G', \omega_{G'}) \geq 2$ then there exist a decomposition $C = E + \Gamma + G_1 + G_2$ s.t.

(a) $G_2 + \Gamma$ *is 4-connected*

(b) $h^1(G_1, \omega_{G_1}) = 1$

(c) $\mathcal{O}_{G_2}(-G_1)$ *is NEF on* G_2

(d) $H^0(G, \omega_G) \twoheadrightarrow H^0(G_2 + \Gamma, \omega_G)$.

Proof. Let $C = E + G$ and $G = \Gamma + G'$ be as in Lemma 5.2. By (e) of Lemma 5.2 G is 4-connected and by our hypothesis $h^1(G', \omega_{G'}) \geq 2$, i.e., the irreducible curve Γ is a disconnecting component for G too. Therefore by Lemma 5.2 applied to G, there exists a maximal connected subcurve $G_1 \subset G'$ and a decomposition $G = \Gamma + G_1 + G_2$ such that (a), (b), (c), (d) hold. $\square$

Proof of Thm. 5.3. The proof of theorem 5.3 will be treated considering separately the case $h^1(G', \omega_{G'}) = 1$ and $h^1(G', \omega_{G'}) \geq 2$.

Case 1: *There exists a disconnecting component Γ and a decomposition $C = G' + \Gamma + E$ as in Lemma 5.2 such that $h^1(G', \omega_{G'}) = 1$.*

Let $G = G' + \Gamma$. On Γ both the invertible sheaves $\omega_\Gamma(E)$ and ω_G have degree $\geq 2p_a(\Gamma)+2$. In particular we have the following exact sequence

$$0 \to H^0(\Gamma, \omega_\Gamma(E)) \to H^0(G, \omega_C) \to H^0(G', \omega_C) \to 0$$

Twisting with $H^0(\omega_G) = H^0(G, \omega_G)$ we get the following commutative diagram:

$$\begin{array}{ccccc} H^0(\Gamma, \omega_\Gamma(E)) \otimes H^0(\omega_G) & \hookrightarrow & H^0(G, \omega_C) \otimes H^0(\omega_G) & \twoheadrightarrow & H^0(G', \omega_C) \otimes H^0(\omega_G) \\ r_1 \downarrow & & r_2 \downarrow & & r_3 \downarrow \\ H^0(\Gamma, \omega_\Gamma(E) \otimes \omega_G) & \hookrightarrow & H^0(G, \omega_C \otimes \omega_G) & \twoheadrightarrow & H^0(G', \omega_C \otimes \omega_G) \end{array}$$

Now, since by our hypothesis $h^1(G', \omega_{G'}) = 1$ then $H^0(G, \omega_G) \twoheadrightarrow H^0(\Gamma, \omega_G)$ and by [17, Thm.6] we have the surjection $H^0(\Gamma, \omega_\Gamma(E)) \otimes H^0(\Gamma, \omega_G) \twoheadrightarrow H^0(\Gamma, \omega_\Gamma(E) \otimes \omega_G)$.

The proposition follows since also r_3 is surjective by Prop. 2.9. Indeed, it is $\omega_{G|G'} \cong \omega_{C|G'}(-E)$ with $\mathscr{O}_{G'}(-E)$ NEF and $\omega_{C|G'}$ is an even invertible sheaf whose degree on every subcurve $B \subseteq G'$ satisfies $\deg_B(\omega_C) \geq 2p_a(B)+2$.

Case 2: *There exists a disconnecting component* Γ *and a decomposition* $C = G' + \Gamma + E$ *as in Lemma 5.2 such that* $h^1(G', \omega_{G'}) \geq 2$.

Let $C = E + \Gamma + G_1 + G_2$ and $G = \Gamma + G_1 + G_2$, be a decomposition as in Lemma 5.4. We proceed as in Case 1, considering the curve $G_2 + \Gamma$ instead of the irreducible Γ.

First of all let us prove that $H^1(G_2 + \Gamma, \omega_{G_2+\Gamma}(E)) = 0$.

It is $(\omega_{G_2+\Gamma}(E))_{|G_2} \cong (\omega_C(-G_1))_{|G_2}$ and in particular for every subcurve $B \subseteq G_2$ it is $\deg_B(\omega_{G_2+\Gamma}(E)) \geq 2p_a(B)+2$ since $\mathscr{O}_{G_2}(-G_1)$ is NEF.

If $B \not\subseteq G_2$, we can write $B = B' + \Gamma$, with $B' \subseteq G_2$, obtaining

$$\deg_B(\omega_{G_2+\Gamma}(E)) = \deg_B(\omega_{G_2+\Gamma}) + E \cdot B \geq 2p_a(B)+2$$

since $E \cdot B = E \cdot (B' + \Gamma) \geq E \cdot (G' + \Gamma) \geq 4$ and $\deg_B(\omega_{G_2+\Gamma}) \geq 2p_a(B) - 2$ by connectedness. Therefore $H^1(G_2 + \Gamma, \omega_{G_2+\Gamma}(E)) = 0$ by [5, Lemma 2.1] and we have the following exact sequence

$$0 \to H^0(G_2 + \Gamma, \omega_{G_2+\Gamma}(E)) \to H^0(G, \omega_C) \to H^0(G_1, \omega_C) \to 0$$

Twisting with $H^0(\omega_G) = H^0(G, \omega_G)$ we can argue as in Case 1. Indeed, consider the commutative diagram:

$$\begin{array}{ccccc} H^0(G_2+\Gamma, \omega_{G_2+\Gamma}(E)) \otimes H^0(\omega_G) & \hookrightarrow & H^0(G, \omega_C) \otimes H^0(\omega_G) & \twoheadrightarrow & H^0(G_1, \omega_C) \otimes H^0(\omega_G) \\ r_1 \downarrow & & r_2 \downarrow & & r_3 \downarrow \\ H^0(G_2+\Gamma, \omega_{G_2+\Gamma}(E) \otimes \omega_G) & \hookrightarrow & H^0(G, \omega_C \otimes \omega_G) & \twoheadrightarrow & H^0(G_1, \omega_C \otimes \omega_G) \end{array}$$

The map r_3 is onto by Prop. 2.9 since $\omega_G \cong \omega_C(-E)$, $\mathscr{O}_{G_1}(-E)$ is NEF and by Lemma 5.4 we have the surjection $H^0(\omega_G) \twoheadrightarrow H^0(G_1, \omega_G)$.

The Theorem follows if we show that r_1 is surjective too. Notice that we can write the multiplication map r_1 as follows

$$r_1 : H^0(G_2 + \Gamma, \omega_{G_2+\Gamma}(E)) \otimes H^0(G_2 + \Gamma, \omega_{G_2+\Gamma}(G_1)) \to H^0(G_2 + \Gamma, \omega_{G_2+\Gamma}^{\otimes 2}(E + G_1))$$

that is, r_1 is symmetric in E and G_1.

Assume firstly that $(G_1 - E)\cdot\Gamma \geq 0$. We proceed considering a general effective Cartier divisor Υ on $G_2+\Gamma$ such that

$$\begin{cases} (\mathscr{O}_{G_2}(\Upsilon))^{\otimes 2} & \overset{\text{num}}{\sim} & \omega_C(-2E)_{|G_2} \\ \deg(\mathscr{O}_\Gamma(\Upsilon)) & = & \frac{1}{2}\deg(\omega_{C|\Gamma}) - E\cdot\Gamma - \delta \end{cases}$$

with $\delta = \lceil \frac{-G_1\cdot G_2}{2} \rceil$. We remark that by connectedness of C and nefness of $\mathscr{O}_{G_2}(-2E)$ it is $\deg(\mathscr{O}_{\Gamma_i}(\Upsilon)) \geq 1$ for every $\Gamma_i \subseteq G_2$ and by our numerical conditions it is

$$\deg(\mathscr{O}_\Gamma(\Upsilon)) = p_a(\Gamma) - 1 + \frac{1}{2}(G_1 - E)\cdot\Gamma + \frac{1}{2}G_2\cdot\Gamma - \delta \geq 1$$

since $(G_1-E)\cdot\Gamma \geq 0$ by our assumptions and $G_2\cdot\Gamma - 2\delta \geq (\Gamma+G_1)\cdot G_2 \geq (\Gamma+G_1+E)\cdot G_2 \geq 4$ by 4-connectedness of C.

Now let $\mathscr{F} := \omega_G(-\Upsilon)$. $\mathscr{F}$ is a general invertible subsheaf of $\omega_{G|G_2+\Gamma}$ s.t.

$$\begin{cases} \deg_B \mathscr{F} & = & \frac{1}{2}\deg_B \omega_C \ \ \forall B \subseteq G_2 \\ \deg_\Gamma \mathscr{F} & = & \frac{1}{2}\deg_\Gamma \omega_C + \delta \end{cases}$$

By 4-connectedness on every subcurve $B \subseteq G_2+\Gamma$ it is $\deg_B \mathscr{F} \geq p_a(B)+1$. By Theorem 2.1 we conclude that $|\mathscr{F}|$ is base point free and $h^1(G_2+\Gamma,\mathscr{F}) = 0$.

Therefore we have the following exact sequence

$$0 \to H^0(G_2+\Gamma,\mathscr{F}) \to H^0(G_2+\Gamma,\omega_G) \to H^0(\mathscr{O}_\Upsilon) \to 0$$

and we have the surjection $H^0(\mathscr{O}_\Upsilon)\otimes H^0(G,\omega_{G_2+\Gamma}(E)) \twoheadrightarrow H^0(G,\mathscr{O}_\Upsilon\otimes\omega_{G_2+\Gamma}(E))$ since $\mathscr{O}_\Upsilon$ is a skyscraper sheaf and $|\omega_{G_2+\Gamma}(E)|$ is base point free by [6, Thm.3.3]. Whence the map r_1 is surjective if we prove that

$$H^0(G_2+\Gamma,\omega_{G_2+\Gamma}(E))\otimes H^0(G_2+\Gamma,\mathscr{F}) \twoheadrightarrow H^0(G_2+\Gamma,\omega_{G_2+\Gamma}(E)\otimes\mathscr{F})$$

To this aim we are going to apply (*iii*) of Prop. 2.5. First notice that

$$H^0(G_2+\Gamma,\mathscr{F}) \hookrightarrow H^0(\Gamma,\mathscr{F})$$

Indeed, by adjunction and Serre duality the kernel of this map is isomorphic to $H^0(G_2,\mathscr{F}-\Gamma) \cong H^1(G_2,\omega_C(-E-G_1)\otimes\mathscr{F}^{-1})$, which vanishes by Thm. 2.1 since it is the first cohomology group of a general invertible sheaf whose degree on every component $B \subseteq G_2$ satisfies

$$\deg_B(\omega_C(-E-G_1)\otimes\mathscr{F}^{-1}) = \frac{1}{2}(\deg_B \omega_C) + (-E-G_1)\cdot B \geq \frac{1}{2}(\deg_B(\omega_C)) \geq p_a(B)$$

because C is 4-connected and $\mathscr{O}_{G_2}(-E-G_1)$ is NEF.

Moreover we have also the embedding

$$H^0(G_2+\Gamma,\omega_{G_2+\Gamma}\otimes[\omega_{G_2+\Gamma}(E)]^{-1}\otimes\mathscr{F}) \hookrightarrow H^0(\Gamma,\omega_{G_2+\Gamma}\otimes[\omega_{G_2+\Gamma}(E)]^{-1}\otimes\mathscr{F})$$

since $H^0(G_2, \mathscr{F}-E-\Gamma) \cong H^1(G_2, \omega_C(-G_1) \otimes \mathscr{F}^{-1}) = 0$ because

$$\deg_B(\omega_C(-G_1) \otimes \mathscr{F}^{-1}) = \frac{1}{2}(\deg_B \omega_C) + (-G_1 \cdot B) \geq \frac{1}{2}(\deg_B(\omega_C)) \geq p_a(B)$$

by 4-connectedness of C and nefness of $\mathscr{O}_{G_2}(-G_1)$.

In order to conclude we are left to compute $h^0(G_2+\Gamma, \omega_{G_2+\Gamma} \otimes [\omega_{G_2+\Gamma}(E)]^{-1} \otimes \mathscr{F}) = h^0(G_2+\Gamma, \mathscr{F}(-E))$.

$\mathscr{F}(-E)$ is a general invertible sheaf s.t.

$$\begin{cases} \deg_B(\mathscr{F}(-E)) & = & \frac{1}{2}\deg_B \omega_C - E \cdot B \qquad \forall B \subseteq G_2 \\ \deg_\Gamma(\mathscr{F}(-E)) & = & \frac{1}{2}\deg_\Gamma(\omega_{G_2+\Gamma}) + \frac{1}{2}(G_1 - E) \cdot \Gamma + \delta \end{cases}$$

Therefore we obtain immediately that $\deg_B \mathscr{F}(-E) \geq p_a(B)$ on every $B \subseteq G_2$, whereas if $B = B' + \Gamma$ with $B' \subseteq G_2$ it is

$$\deg_B(\mathscr{F}(-E)) = \frac{1}{2}\deg_B(\omega_{G_2+\Gamma}) + \frac{1}{2}(G_1 - E) \cdot B' + \frac{1}{2}(G_1 - E) \cdot \Gamma + \delta \geq p_a(B)$$

since by our assumptions $G_2 + \Gamma$ is numerically connected, $(G_1 - E) \cdot \Gamma \geq 0$, $E \cdot B' \leq 0$ and $\delta \geq \frac{1}{2}(-G_1 \cdot G_2) \geq \frac{1}{2}(-G_1 \cdot B')$.

In particular by Theorem 2.1 we get $H^1(G_2+\Gamma, \mathscr{F}(-E)) = 0$ and by Riemann-Roch theorem we have $h^0(G_2+\Gamma, \mathscr{F}(-E)) = h^0(G_2+\Gamma, \mathscr{F}) - E \cdot (\Gamma + G_2)$. Finally, since $\mathscr{O}_{G_1}(-E)$ is NEF we have $E \cdot (\Gamma + G_2) \geq E \cdot (\Gamma + G_2 + G_1) \geq 4$ because C is 4-connected, that is, $h^0(G_2+\Gamma, \mathscr{F}(-E)) \leq h^0(G_2+\Gamma, \mathscr{F}) - 4$. Whence all the hypotheses of (*iii*) of Prop. 2.5 are satisfied and we can conclude.

If $(E - G_1) \cdot \Gamma < 0$ we exchange the role of $\mathscr{O}_{G_2+\Gamma}(E)$ with the one of $\mathscr{O}_{G_2+\Gamma}(G_1)$ and we reply the proof "verbatim", since our numerical conditions are symmetric in E and G_1.

$\square$

6 The canonical ring of an even 4-connected curve

In this section we are going to show Theorem 1.1.

Proof of Theorem 1.1. For all $k \in \mathbb{N}$ we have to show the surjectivity of the maps

$$\rho_k : (H^0(C, \omega_C))^{\otimes k} \longrightarrow H^0(C, \omega_C^{\otimes k})$$

For $k = 0, 1$ it is obvious. For $k \geq 3$ it follows by an induction argument applying Prop. 2.5 to the sheaves $\omega_C^{\otimes(k-1)}$ and ω_C.

For $k = 2$ the proof is based on the above results. If C is irreducible and reduced the result is (almost) classical. For the general case we separate the proof in three different parts, depending on the existence of suitable irreducible components.

Case A: *There exists a not disconnecting irreducible curve* Γ *of arithmetic genus* $p_a(\Gamma) \geq 1$.

In this case, writing $C=\Gamma+E$, we have the surjections $H^0(C,\omega_C)\twoheadrightarrow H^0(\Gamma,\omega_{C|\Gamma})$ and $H^0(C,\omega_C)\twoheadrightarrow H^0(E,\omega_{C|E})$, whence we can conclude by the following commutative diagram:

$$\begin{array}{ccccc} H^0(\Gamma,\omega_\Gamma)\otimes H^0(\omega_C) & \hookrightarrow & H^0(\omega_C)\otimes H^0(\omega_C) & \twoheadrightarrow & H^0(E,\omega_C)\otimes H^0(\omega_C) \\ r_1\downarrow & & \rho_2\downarrow & & r_3\downarrow \\ H^0(\Gamma,\omega_\Gamma\otimes\omega_C) & \hookrightarrow & H^0(C,\omega_C^{\otimes 2}) & \twoheadrightarrow & H^0(E,\omega_C^{\otimes 2}) \end{array}$$

(where $H^0(\omega_C)=H^0(C,\omega_C)$). Indeed, since C is 4-connected and ω_C is an even divisor we get the surjection of the map r_3 by Theorem 3.1, while Proposition 2.6 ensure the surjectivity of the map r_1, forcing ρ_2 to be surjective too (cf. also [17, Thm. 6]).

Case B: *There exists a disconnecting irreducible component* Γ.

Let us consider the decomposition $C=E+G$ introduced in Lemma 5.2. Then we have the exact sequence

$$0\to H^0(G,\omega_G)\to H^0(C,\omega_C)\to H^0(E,\omega_C)\to 0$$

and furthermore by Lemma 5.2 *(i)* also the map $H^0(C,\omega_C)\to H^0(G,\omega_C)$ is onto. Replacing Γ with G we can build a commutative diagram analogous to the one shown in case A. Keeping the notation r_1,r_3 for the analogous maps, by Theorem 3.1 and Theorem 5.3 the maps r_3 and r_1 are surjectives, whence also ρ_2 is onto.

Case C: *Every irreducible component* Γ_i *of* C *has arithmetic genus* $p_a(\Gamma_i)=0$ *and it is not disconnecting.*

First of all notice that by connectedness for every irreducible Γ_h there exists at least one Γ_k such that $\Gamma_h\cdot\Gamma_k\geq 1$, and that $h^0(B,\mathscr{O}_B)=1$ for a curve $B=\sum a_i\Gamma_i\subset C$ implies $\Gamma_i\cdot(B-\Gamma_i)\geq 1$ for every $\Gamma_i\subset B$.

We will consider separately the different situations that may happen.

C.1. *There exist two components* Γ_h,Γ_k *(possibly* $h=k$ *if* $\mathrm{mult}_C\Gamma_h\geq 2$*) such that* $\Gamma_h\cdot\Gamma_k\geq 2$*, and* $\Gamma=\Gamma_h+\Gamma_k\subset C$ *is not disconnecting.*

Notice that by §4 we may assume $C-\Gamma\neq\emptyset$. In this situation, setting $E=C-\Gamma$ by *(ii)* of Proposition 2.6 we can proceed exactly as in *Case A*.

C.2. *There exist two components* Γ_h,Γ_k *(possibly* $h=k$ *if* $\mathrm{mult}_C\Gamma_h\geq 2$*) such that* $\Gamma:=\Gamma_h+\Gamma_k\subset C$ *is disconnecting and* $\Gamma_h\cdot\Gamma_k\geq 0$.

In this case take E a maximal subcurve of $C-\Gamma_k-\Gamma_h$ such that $h^0(E,\mathscr{O}_E)=1$ and let $G=C-E$. Then we obtain a decomposition $C=E+\Gamma+G'$ with $\mathscr{O}_{G'}(-E)$ NEF on G'.

Firstly let us point out some useful remarks about this decompositions.

We have $h^0(E+\Gamma_k+G',\mathscr{O}_{E+\Gamma_k+G'})=1$ since Γ_k is not disconnecting in C and it is immediately seen that also $h^0(\Gamma_k+G',\mathscr{O}_{\Gamma_h+G'})=1$ because $\mathscr{O}_{G'}(-E)$ is NEF on

G'. But $p_a(\Gamma_k) = 0$, whence by the remark given at the beginning of Case C it is $\Gamma_k \cdot G' \geq 1$. In particular $H^0(G, \omega_G) \twoheadrightarrow H^0(\Gamma_h, \omega_G)$. Similarly we obtain $\Gamma_h \cdot G' \geq 1$, $H^0(G, \omega_G) \twoheadrightarrow H^0(\Gamma_k, \omega_G)$, and considering $\mathscr{O}_\Gamma(E)$ we have $E \cdot \Gamma_h \geq 1$ and $E \cdot \Gamma_k \geq 1$. Furthermore, since $E \cdot \Gamma \geq 4$, may assume $E \cdot \Gamma_h \geq 2$.

We will consider firstly the subcase where $\Gamma_h \cdot \Gamma_k \geq 1$ and secondly the case where the product is null.

C.2.1. If $\Gamma_h \cdot \Gamma_k \geq 1$ and $\Gamma = \Gamma_h + \Gamma_k \subset C$ is disconnecting, arguing as in *Case B*, the theorem follows if we have the surjection of the multiplication map $r_1 : H^0(G, \omega_C) \otimes H^0(G, \omega_G) \to H^0(G, \omega_C \otimes \omega_G)$. Considering the diagram

$$\begin{array}{ccccc} H^0(\Gamma, \omega_\Gamma(E)) \otimes H^0(\omega_G) & \hookrightarrow & H^0(G, \omega_C) \otimes H^0(\omega_G) & \twoheadrightarrow & H^0(G', \omega_C) \otimes H^0(\omega_G) \\ s_1 \downarrow & & r_1 \downarrow & & t_1 \downarrow \\ H^0(\Gamma, \omega_\Gamma(E) \otimes \omega_G) & \hookrightarrow & H^0(G, \omega_C \otimes \omega_G) & \twoheadrightarrow & H^0(G', \omega_C \otimes \omega_G) \end{array}$$

it is sufficient to show that s_1 is onto since t_1 is surjective by Prop. 2.9. To this aim we take the splitting

$$0 \to H^0(\Gamma_h, \omega_{\Gamma_h}(E)) \to H^0(\Gamma, \omega_\Gamma(E)) \to H^0(\Gamma_k, \omega_{\Gamma_k}(\Gamma_h + E)) \to 0$$

Twisting with $H^0(G, \omega_G) = H^0(\omega_G)$ (notice that $H^0(G, \omega_G) \twoheadrightarrow H^0(\Gamma_h, \omega_G)$ and similarly for Γ_k by the above remark), we can conclude since we have the surjections

$$H^0(\Gamma_h, \omega_{\Gamma_h}(E)) \otimes H^0(\Gamma_h, \omega_G) \twoheadrightarrow H^0(\Gamma_h, \omega_{\Gamma_h}(E) \otimes \omega_G)$$
$$H^0(\Gamma_k, \omega_{\Gamma_k}(\Gamma_h + E)) \otimes H^0(\Gamma_k, \omega_G) \twoheadrightarrow H^0(\Gamma_k, \omega_{\Gamma_k}(\Gamma_h + E) \otimes \omega_G)$$

because $\Gamma_h \cong \Gamma_k \cong \mathbb{P}^1$, and all the sheaves have positive degree on both the curves (see [12, Corollary 3.a.6] for details).

C.2.2. Assume now $\Gamma_h \cdot \Gamma_k = 0$ and $\Gamma = \Gamma_h + \Gamma_k \subset C$ to be disconnecting.

If $E \cdot \Gamma_h \geq 2$, $E \cdot \Gamma_k \geq 2$ and $G' \cdot \Gamma_h \geq 2$, $G' \cdot \Gamma_k \geq 2$ then we consider the exact sequence

$$0 \to \omega_\Gamma(E) \to \omega_{C|(G'+\Gamma)} \to \omega_{C|G'} \to 0$$

and we operate as in *C.2.1*.

Otherwise, without loss of generality, we may assume $E \cdot \Gamma_h = 1$ or $G' \cdot \Gamma_h = 1$.

If $E \cdot \Gamma_h = 1$ then by 4-connectedness of C it is $E \cdot \Gamma_k \geq 3$, $G' \cdot \Gamma_h \geq 3$, $G' \cdot \Gamma_k \geq 3$.

Let $G = G' + \Gamma_h + \Gamma_k$ and consider the splitting $C = E + G$: as in the previous case it is enough to prove the surjection of $r_1 : H^0(G, \omega_C) \otimes H^0(G, \omega_G) \to H^0(G, \omega_C \otimes \omega_G)$.

To this aim we take the following exact sequence

$$0 \to H^0(\Gamma_k, \omega_{\Gamma_k}(E)) \to H^0(G, \omega_C) \to H^0(G' + \Gamma_h, \omega_C)$$

By our numerical conditions $|\omega_G|$ is base point free and by connectedness $H^0(\omega_G) \twoheadrightarrow H^0(\Gamma_k, \omega_G)$.

By [12, (2.a.17), (3.a.6)] (or simply since we have sheaves of positive degree on a rational curve) we have the surjection $H^0(\Gamma_k,\omega_G)\otimes H^0(\Gamma_k,\omega_{\Gamma_k}(E))\twoheadrightarrow H^0(\Gamma_k,\omega_G\otimes\omega_{\Gamma_k}(E))$.

On the contrary it is $\deg\mathscr{O}_{G'+\Gamma_h}(-E)\geq -1$. Therefore we can consider a subsheaf $\mathscr{F}\subset\omega_{C|G'+\Gamma_h}$ such that $(\mathscr{F}_{|G'+\Gamma_h})^{\otimes 2}\overset{\text{num}}{\sim}\omega_{C|G'+\Gamma_h}$. Then for every $B\subset G'+\Gamma_h$ $\mathscr{F}_{|B}$ has degree at least $p_a(B)+1$ whilst $\omega_G\otimes\mathscr{F}^{-1}$ is an invertible sheaf of degree at least $p_a(B)$. Whence by Prop. 2.9 $H^0(G'+\Gamma_h,\omega_G)\otimes H^0(G'+\Gamma_h,\omega_C)\twoheadrightarrow H^0(G'+\Gamma_h,\omega_G\otimes\omega_C)$ and then r_1 is onto.

If $G'\cdot\Gamma_h=1$ then by 4-connectedness of C it is $E\cdot\Gamma_h\geq 3$, $E\cdot\Gamma_k\geq 3$, $G'\cdot\Gamma_k\geq 3$.

In this case we write $\tilde{E}:=E+\Gamma_h$, $\tilde{G}:=G'+\Gamma_k$. We have a decomposition $C=\tilde{E}+\tilde{G}$ where $\tilde{E}$ is connected, $\tilde{G}$ is 3-connected. Moreover by adjunction we have the isomorphism $\omega_{\tilde{G}}=\omega_C(-\tilde{E})_{|\tilde{G}}$, where $\deg\mathscr{O}_{\tilde{G}}(-\tilde{E})\geq -1$.

Arguing as above the theorem follows if we prove the surjection of $\tilde{r}_1: H^0(\tilde{G},\omega_C)\otimes H^0(\tilde{G},\omega_{\tilde{G}})\to H^0(G,\omega_C\otimes\omega_{\tilde{G}})$.

To this aim let us show firstly that $h^0(G',\mathscr{O}_{G'})=1$. Indeed, since Γ_k is not disconnecting for C, whilst it is disconnecting for $C-\Gamma_h$ it is $h^0(G'+\Gamma_h,\mathscr{O}_{G'+\Gamma_h})=1$. Therefore since $\deg\mathscr{O}_{\Gamma_h}(-G')=-1$ and $\Gamma_h\cong\mathbb{P}^1$ by the exact sequence

$$0=H^0(\Gamma_h,\mathscr{O}_{\Gamma_h}(-G'))\to H^0(G'+\Gamma_h,\mathscr{O}_{G'+\Gamma_h})\to H^0(G',\mathscr{O}_{G'})\to H^1(\Gamma_h,\mathscr{O}_{\Gamma_h}(-G'))=0$$

we obtain $h^0(G',\mathscr{O}_{G'})=1$. Going back to $\tilde{G}:=G'+\Gamma_k$ let us take the exact sequence

$$0\to H^0(\Gamma_k,\omega_{\Gamma_k}(E))\to H^0(\tilde{G},\omega_C)\to H^0(G',\omega_C)\to 0.$$

We have $H^0(\Gamma_k,\omega_{\Gamma_k}(E))\otimes H^0(\Gamma_k,\omega_{\tilde{G}})\twoheadrightarrow H^0(\Gamma_k,\omega_{\Gamma_k}(E)\otimes\omega_{\tilde{G}})$ since $\Gamma_k\cong\mathbb{P}^1$.

Finally, taking a subsheaf $\mathscr{F}\subset\omega_{C|\tilde{G}}$ such that $(\mathscr{F}_{\tilde{G}})^{\otimes 2}\overset{\text{num}}{\sim}\omega_{C|\tilde{G}}$, we can consider the two sheaves $\mathscr{G}_1:=\omega_C\otimes\mathscr{F}^{-1}\overset{\text{num}}{\sim}\mathscr{F}$, $\mathscr{G}_2:=\omega_{\tilde{G}}\otimes\mathscr{F}^{-1}$.

$\mathscr{F},\mathscr{G}_1,\mathscr{G}_2$ satisfy the assumptions of Prop. 2.9, whence $\tilde{r}_1$ is surjective and we can conclude.

C.3. *There exists two distinct irreducible components* Γ_h, Γ_k *such that* $\Gamma_h\cdot\Gamma_k=0$, *and* $\Gamma=\Gamma_h+\Gamma_k$ *is not disconnecting.*

In this case the situation is slightly different.

By §2.3 $\rho_2:(H^0(C,\omega_C))^{\otimes 2}\twoheadrightarrow H^0(C,\omega_C^{\otimes 2})$ iff $\mathscr{K}_{0,1}(C,\omega_C,\omega_C)=0$, and by [12, (2.a.17)] it is $\mathscr{K}_{0,1}(C,\omega_C,\omega_C)=\mathscr{K}_{0,2}(C,\omega_C)$.

Write $C=A+\Gamma_h+\Gamma_k$: since it is $p_a(\Gamma_i)=0$, $p_a(\Gamma_h+\Gamma_k)=-1$ and all these curves are not disconnecting then we can consider the long exact sequence of Koszul groups (Prop. 2.7) for the decomposition $C=A+\Gamma_h+\Gamma_k$ (respectively for the decompositions $C-\Gamma_h=A+\Gamma_k$, $C-\Gamma_k=A+\Gamma_h$).

Let $W=\operatorname{im}\{H^0(\omega_C)\to H^0(A,\omega_C)\}$. By Thm. 3.1 for $i\in\{h,k\}$ and every $q\geq 1$ it is $\mathscr{K}_{0,q}(A+\Gamma_i,\omega_C)=0$; consequently it is $\mathscr{K}_{0,q}(A,W,\omega_C)=0$ $\forall q\geq 1$.

Therefore it is sufficient to prove that the following sequence is exact

$$\mathscr{K}_{1,1}(C,\omega_C)\xrightarrow{\ \iota\ }\mathscr{K}_{1,1}(A,W,\omega_C)\xrightarrow{\ \pi\ }\mathscr{K}_{0,2}(\Gamma_h+\Gamma_k,\mathscr{O}_{\Gamma_h+\Gamma_k}(-A),\omega_C)).$$

By [12, (2.a.17)] and adjunction we have

$$\mathscr{K}_{1,1}(\Gamma_k, \mathscr{O}_{\Gamma_k}(-A-\Gamma_h), \omega_C) = \mathscr{K}_{1,0}(\Gamma_k, \omega_{\Gamma_k}, \omega_C).$$

By [12, (2.a.17), (3.a.6)] we get $\mathscr{K}_{1,0}(\Gamma_k, \omega_{\Gamma_k}, \omega_C) = 0$ since $\Gamma_k \cong \mathbb{P}^1$. Similarly $\mathscr{K}_{1,1}(\Gamma_h, \mathscr{O}_{\Gamma_h}(-A-\Gamma_k), \omega_C) = 0$. Moreover, since $\Gamma_h \cap \Gamma_k = \{\emptyset\}$ it is $\mathscr{O}_{\Gamma_k}(-A-\Gamma_h) \cong \mathscr{O}_{\Gamma_k}(-A)$ and $\mathscr{O}_{\Gamma_h}(-A-\Gamma_k) \cong \mathscr{O}_{\Gamma_h}(-A)$, whence we obtain the splitting of the exact sequence of invertible sheaves

$$0 \to \mathscr{O}_{\Gamma_k}(-A-\Gamma_h) \to \mathscr{O}_{\Gamma_h+\Gamma_k}(-A) \to \mathscr{O}_{\Gamma_h}(-A) \to 0,$$

and for every p, q the isomorphism

$$\mathscr{K}_{p,q}(\Gamma_h+\Gamma_k, \mathscr{O}_{\Gamma_h+\Gamma_k}(-A), \omega_C)) \cong \mathscr{K}_{p,q}(\Gamma_h, \mathscr{O}_{\Gamma_h}(-A), \omega_C) \bigoplus \mathscr{K}_{p,q}(\Gamma_k, \mathscr{O}_{\Gamma_k}(-A), \omega_C))$$

In particular we get $\mathscr{K}_{1,1}(\Gamma_h+\Gamma_k, \mathscr{O}_{\Gamma_h+\Gamma_k}(-A), \omega_C)) = 0$, that is ι is injective. To prove the surjectivity of π we consider the following commutative diagram:

$$\begin{tikzcd}
\mathscr{K}_{1,1}(C,\omega_C) \arrow[r, hook] \arrow[d, hook] & \mathscr{K}_{1,1}(A+\Gamma_h,\omega_C) \arrow[d, hook] \arrow[dr] & \\
\mathscr{K}_{1,1}(A+\Gamma_k,\omega_C) \arrow[r, hook] \arrow[dr] & \mathscr{K}_{1,1}(A,W,\omega_C) \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] \arrow[dr, "\pi"] & \mathscr{K}_{0,2}(\Gamma_k,\mathscr{O}_{\Gamma_k}(-A),\omega_C) \arrow[d] \\
 & \mathscr{K}_{0,2}(\Gamma_h,\mathscr{O}_{\Gamma_h}(-A),\omega_C) \arrow[r] & \mathscr{K}_{0,2}(\Gamma_h,\mathscr{O}_{\Gamma_h}(-A),\omega_C) \oplus \mathscr{K}_{0,2}(\Gamma_k,\mathscr{O}_{\Gamma_k}(-A),\omega_C)
\end{tikzcd}$$

Now p is surjective since $\mathscr{K}_{0,2}(A+\Gamma_k, \omega_C) = 0$. Analogously q is surjective. Moreover we have $\pi = (\pi_1, \pi_2)$, and $\mathscr{K}_{1,1}(C, \omega_C) = \mathscr{K}_{1,1}(A+\Gamma_h, \omega_C) \cap \mathscr{K}_{1,1}(A+\Gamma_k, \omega_C)$ (that is the space of quadrics vanishing along C is given considering the intersection of the quadrics vanishing along $A+\Gamma_h$, resp. along $A+\Gamma_h$). Therefore π is surjective, which implies $\mathscr{K}_{0,2}(C, \omega_C) = 0$.

C.4. *For every irreducible subcurve Γ_i it is $\Gamma_i^2 \leq 1$, and for for every distinct pairs Γ_i, Γ_j it is $\Gamma_i \cdot \Gamma_j = 1$; moreover the curve $(\Gamma_i + \Gamma_j) \subset C$ is always not disconnecting.*

C.4.1. Assume that C contains three components $\Gamma_1, \Gamma_2, \Gamma_3$ (possibly equal) such that $\Gamma_1 \cdot \Gamma_2 = \Gamma_2 \cdot \Gamma_3 = \Gamma_1 \cdot \Gamma_3 = 1$ and $\Gamma := \Gamma_1 + \Gamma_2 + \Gamma_3$ is not disconnecting. (Notice that if C has only one irreducible component, then we are exactly in this case since necessarily it is $\Gamma_1^2 = 1$ and $\mathrm{mult}_C(\Gamma_1) \geq 5$).

In this case Γ is 2-connected with arithmetic genus =1, $E = C - G \neq \emptyset$ and then we can proceed as in *Case A*, since it is $\omega_\Gamma \cong \mathscr{O}_\Gamma$.

C.4.2. Assume that C contains three components $\Gamma_1, \Gamma_2, \Gamma_3$ (possibly equal) such that $\Gamma_1 \cdot \Gamma_2 = \Gamma_2 \cdot \Gamma_3 = \Gamma_1 \cdot \Gamma_3 = 1$ and the curve $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3$ is disconnecting.

In this case we can write $C - \Gamma_1 - \Gamma_2 = E + \Gamma_3 + G'$ with E, G' as in Lemma 5.2, that is, we have a decomposition $C = E + \Gamma + G'$ with $\mathscr{O}_{G'}(-E)$ NEF. Moreover it is $E \cdot \Gamma_3 \geq 1$, $G' \cdot \Gamma_3 \geq 1$ since $\Gamma_3 \cong \mathbb{P}^1$ and $\Gamma_1 + \Gamma_2$ is not disconnecting, and similar inequalities hold for Γ_1 and Γ_2.

Let $G = G' + \Gamma$. Since E is connected It is enough to prove that $r_1 : H^0(G, \omega_C \otimes H^0(G, \omega_G) \to H^0(G, \omega_C \otimes \omega_G)$.

Notice that for every $i \in \{1,2,3\}$ we have $\deg_{\Gamma_i} \omega_G \geq 0$ and $H^0(G, \omega_G) \twoheadrightarrow H^0(\Gamma_i, \omega_G)$.

Without loss of generality we may assume $E \cdot \Gamma_1 \geq 2$ since $E \cdot (\Gamma_1 + \Gamma_2 + \Gamma_3) \geq 4$. We work as in *Case C.2.1*, i.e., we consider the splitting $G = \Gamma + G'$ and we take the sheaf $\omega_\Gamma(E)$. Then we have the exact sequence

$$0 \to H^0(\Gamma_1, \omega_{\Gamma_1}(E)) \to H^0(\Gamma, \omega_\Gamma(E)) \to H^0(\Gamma_2 + \Gamma_3, \omega_{\Gamma_2+\Gamma_3}(\Gamma_1 + E)) \to 0$$

and by the same degree arguments adopted in *Case C.2.1* it is immediately seen that we have the surjective maps

$$H^0(\Gamma_1, \omega_{\Gamma_1}(E)) \otimes H^0(\Gamma_1, \omega_G) \twoheadrightarrow H^0(\Gamma_1, \omega_{\Gamma_1}(E) \otimes \omega_G)$$

$$H^0(\Gamma_2+\Gamma_3, \omega_{\Gamma_2+\Gamma_3}(\Gamma_1+E)) \otimes H^0(\Gamma_2+\Gamma_3, \omega_G) \twoheadrightarrow H^0(\Gamma_2+\Gamma_3, \omega_{\Gamma_2+\Gamma_3}(\Gamma_1+E) \otimes \omega_G)$$

that is, we get the surjection $H^0(\Gamma, \omega_\Gamma(E)) \otimes H^0(\omega_G) \twoheadrightarrow H^0(\Gamma, \omega_\Gamma(E) \otimes \omega_G)$. Finally, as in the previous cases $H^0(G, \omega_C) \otimes H^0(\omega_G) \twoheadrightarrow H^0(G, \omega_C \otimes \omega_G)$ since $\mathcal{O}_{G'}(-E)$ is NEF, and then we can conclude that r_1 is onto.

C.4.3. Finally, we are left with the case where C has exactly two irreducible components, Γ_1, Γ_2 of nonpositive selfintersection: $C = n_1\Gamma_1 + n_2\Gamma_2$, $\Gamma_1 \cdot \Gamma_2 = 1$ and $\Gamma_i^2 \leq 0$ for $i = 1, 2$.

We may assume $\Gamma_1^2 \geq \Gamma_2^2$. Since C is 4-connected with an easy computation we obtain $\Gamma_1^2 = 0$ and $\mathrm{mult}_C(\Gamma_i) \geq 4$. Moreover n_2 is even since C is an even curve.

Notice that for every subcurve $B = \alpha_1\Gamma_1 + \alpha_2\Gamma_2 \subset C$ it is

$$B \cdot (C-B) = \alpha_2(n_1 + (n_2-1)\Gamma_2^2) - \alpha_2(\alpha_2-1)\Gamma_2^2 + \alpha_1(n_2 - 2\alpha_2) \geq \alpha_2(n_1 + (n_2-1)\Gamma_2^2)$$

(since we may assume $2\alpha_2 \leq n_2$ by the symmetry of the intersection product), which implies $B \cdot (C-B) \geq 4\alpha_2$ since $\Gamma_2 \cdot (C - \Gamma_2) = n_1 + (n_2-1)\Gamma_2^2 \geq 4$.

If $\Gamma_2^2 = 0$, we take $E = 2\Gamma_1 + 2\Gamma_2$. Then $p_a(E) = 1$ and applying a refinement of the above formula it is easy to see that $C - E$ is numerically connected. In this case we can conclude as in *Case A*.

If $\Gamma_2^2 < 0$, let $a_1 = \lceil \frac{n_1}{2} \rceil$, $a_2 = \frac{n_2}{2}$ and let $G := a_1\Gamma_1 + a_2\Gamma_2$, $E := C - G = (n_1 - a_1)\Gamma_1 + a_2\Gamma_2$.

Now E is numerically connected and G is 2-connected. Indeed, let us consider a subcurve $B = \alpha_1\Gamma_1 + \alpha_2\Gamma_2 \subset G$. Since $2G \cdot B \geq G \cdot B$ we have $B \cdot (G-B) \geq \frac{1}{4} 2B \cdot (C - 2B) \geq 2$ since $2B \cdot (C - 2B) \geq 8\alpha_2$ by the above formula. If $B \subset E$ it is $B \cdot (E - B) \geq \frac{1}{4} 2B \cdot (C - \Gamma_1 - 2B) \geq 1$.

Therefore it is enough to prove the surjection of $r_1 : H^0(G, \omega_C) \otimes H^0(G, \omega_G) \to H^0(G, \omega_G \otimes \omega_C)$. We have the following exact sequence

$$0 \to H^0(a_1\Gamma_1 + \Gamma_2, \omega_{a_1\Gamma_1+\Gamma_2}(E)) \to H^0(G, \omega_C) \to H^0((a_2-1)\Gamma_2, \omega_C) \to 0$$

and moreover $H^0(G, \omega_G) \twoheadrightarrow H^0((a_2-1)\Gamma_2, \omega_G)$ since $a_1\Gamma_1 + \Gamma_2$ is numerically connected.

By [6, Thms. 3.3] $|\omega_G|$ is a base point free system on G since G is 2-connected. Let $W := \mathrm{im}\{H^0(G,\omega_G) \to H^0(a_1\Gamma_1+\Gamma_2,\omega_G)\}$. Then W is a base point free system and moreover we have $H^1(a_1\Gamma_1+\Gamma_2,\omega_{a_1\Gamma_1}(E)\otimes{\omega_G}^{-1}) \cong H^1(a_1\Gamma_1+\Gamma_2, E-(a_2-1)\Gamma_2) = 0$ because $\Gamma_i \cong \mathbb{P}^1$, $\Gamma_1^2 = 0$, $\Gamma_1\cdot\Gamma_2 = 1$ and $(E-(a_2-1)\Gamma_2)\cdot\Gamma_1 = 1$, $(E-(a_2-1)\Gamma_2)\cdot\Gamma_2 \geq 1$ since E is 1-connected. Therefore by Prop. 2.5 we have the surjection

$$H^0(a_1\Gamma_1+\Gamma_2,\omega_{a_1\Gamma_1+\Gamma_2}(E))\otimes W \twoheadrightarrow H^0(a_1\Gamma_1+\Gamma_2,\omega_{a_1\Gamma_1+\Gamma_2}(E)\otimes\omega_G)$$

Finally $H^0((a_2-1)\Gamma_2,\omega_G)\otimes H^0((a_2-1)\Gamma_2,\omega_C) \twoheadrightarrow H^0((a_2-1)\Gamma_2,\omega_G\otimes\omega_C)$ follows from (i) of Prop. 2.5 taking $\mathscr{H} = \omega_C, \mathscr{F} = \omega_G$ if $\mathscr{O}_{\Gamma_2}(E)$ is NEF, or $\mathscr{F} = \omega_C, \mathscr{H} = \omega_G$ if $\mathscr{O}_{\Gamma_2}(-E)$ is NEF.

Q.E.D. for Theorem 1.1

7 On the canonical ring of regular surfaces

In this section we prove Theorem 1.2. The arguments we adopt are very classical and based on the ideas developed in [8]. Essentially we simply restrict to a curve in the canonical system $|K_S|$. The only novelty is that now we do not make any requests on such a curve (i.e. we allow the curve $C \in |K_S|$ to be singular and with many components) since we can apply Thm. 1.1.

Proof of Theorem 1.2.

By assumption there exists a 3-connected not honestly hyperelliptic curve $C = \sum_{i=1}^s n_i\Gamma_i \in |K_S|$. Let $s \in H^0(S,K_S)$ be the corresponding section, so that C is defined by $(s) = 0$.

By adjunction we have $(K_S^{\otimes 2})_{|C} = (K_S+C)_{|C} \cong \omega_C$, that is, C is an even curve; in particular it is 4-connected. Thus we can apply Theorem 1.1, obtaining the surjection

$$(H^0(C,K_S^{\otimes 2}))^{\otimes k} \twoheadrightarrow H^0(C,K_S^{\otimes 2k}) \quad \forall k \in \mathbb{N}.$$

Now let us consider the usual maps given by multiplication of sections

$$A_{l,m} : H^0(S,K_S^{\otimes l})\otimes H^0(S,K_S^{\otimes m}) \to H^0(S,K_S^{\otimes(l+m)})$$
$$a_{l,m} : H^0(C,K_S^{\otimes l})\otimes H^0(C,K_S^{\otimes m}) \to H^0(C,K_S^{\otimes(l+m)})$$

and consider the following commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] \\
 & H^0(S,K_S^{\otimes(k-1)}) \arrow[r, "\cong"] \arrow[d, "C_k"] & H^0(S,K_S^{\otimes(k-1)}) \arrow[d, "c_k"] \\
\bigoplus\limits_{\substack{l+m=k\\0<l\leq m}} & \left[H^0(S,K_S^{\otimes l})\otimes H^0(S,K_S^{\otimes m})\right] \arrow[r, "\overline{\rho}_k"] \arrow[d, "R_k"] & H^0(S,K_S^{\otimes k}) \arrow[d, "r_k"] \\
\bigoplus\limits_{\substack{l+m=k\\0<l\leq m}} & \left[H^0(C,K_S^{\otimes l})\otimes H^0(C,K_S^{\otimes m})\right] \arrow[r, "\rho_k"] \arrow[d] & H^0(C,K_S^{\otimes k}) \arrow[d] \\
 & 0 & 0
\end{tikzcd}$$

Here the left hand column is a complex, while the right hand column is exact. Moreover

- C_k is given by tensor product with s while c_k is given by product with s
- $R_k = \bigoplus\limits_{\substack{l+m=k\\0<l\leq m}} r_l \otimes r_m$ (where r_l, r_m are the usual restrictions)
- $\overline{\rho}_k = \bigoplus\limits_{\substack{l+m=k\\0<l\leq m}} A_{l,m}$ and $\rho_k = \bigoplus\limits_{\substack{l+m=k\\0<l\leq m}} a_{l,m}$

Note that it is $\mathrm{coker}(\rho_k) \cong \mathrm{coker}(\overline{\rho}_k)$ for every $k \in \mathbb{N}$.

Now, for S of general type, if $p_g \geq 1$ and $q = 0$ by [8, Thm. 3.4] $\overline{\rho}_k$ is surjective for every $k \geq 5$ except the case $p_g = 2,\ K^2 = 1$, which is not our case since otherwise C would be a curve of genus 2 contradicting our assumptions. For $k = 4$ the map $a_{2,2}$ is surjective by Thm. 1.1. Whence ρ_4 and $\overline{\rho}_4$ are surjective, and this proves the theorem.

Q.E.D. for Theorem 1.2.

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Author's address:

Marco Franciosi
Dipartimento di Matemativa Applicata "U. Dini", Università di Pisa
via Buonarroti 1C, I-56127, Pisa, Italy
e-mail: franciosi@dma.unipi.it